

Situation: Restrictions on Exponential Functions

Prepared at the University of Georgia in Dr. Wilson's EMAT 6500 Class
July 25, 2013 - Sarah Major

Prompt:

A teacher prompts her students to turn in their homework on exponential functions, but one of the students raises his hand and says, "I don't understand why we have restrictions on the bases of exponential functions. You can raise 0 to any power and still get a value, and you can do the same for 1 and negative numbers. So why can't we use those?"

Commentary:

Though this prompt is fairly simple, it has a lot of relevancy within a high school Mathematics classroom. Students are told of these restrictions on the base values of exponential functions, but when they question their teacher on why, many teachers simply state that it is contained in the definition of the function. They do not give intuitive reasons as to why these values cannot be included as bases of exponential functions. This lack of explanation may not only hinder their understanding of the true properties of exponential functions but also the properties of these numbers in general.

Mathematical Foci:

Mathematical Focus 1

Exponential functions adhere to distinct properties, including those that limit the values of what the base can be.

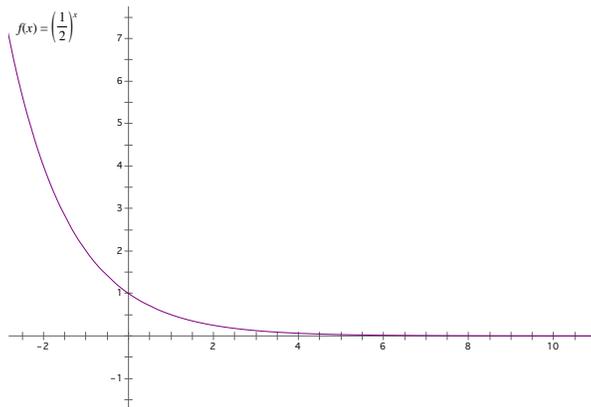
The parent form of the exponential function appears in the form:

$$f(x) = b^x$$

where b , known as the **base** is a fixed number and x is any real number. These types of functions adhere to certain properties:

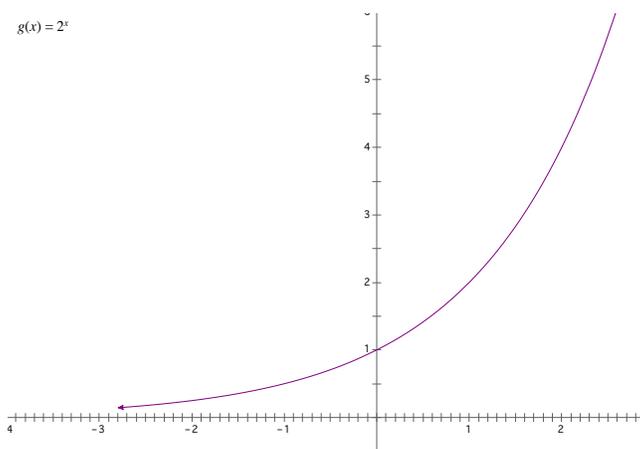
1. The graph of $f(x) = b^x$ will always contain the point (0,1). This is because any number raised to the 0 power will always be 1.
 - Examples: $1^0 = 1$, $2^0 = 1$, $(1/2)^0 = 1$, etc.
2. $b^x > 0$ for every b . Therefore, there is no x intercept.
3. The domain is all real numbers.
4. If $0 < b < 1$, then the graph of $f(x) = b^x$ will decrease as $x \rightarrow \infty$.

- Example: $f(x) = \left(\frac{1}{2}\right)^x$, the graph of which looks like:



Notice that as the other properties state, the graph passes through the point (0,1) and is always greater than 0.

5. If $b > 1$, then the graph of $f(x) = b^x$ will increase as $x \rightarrow \infty$.
 - Example: $f(x) = 2^x$, the graph of which looks like:



Notice once again that the graph passes through the point (0,1) and is always greater than 0.

6. The function is one-to-one. Therefore, if $b^x = b^y$, then $x = y$.

The general form of an exponential function appears in the form:

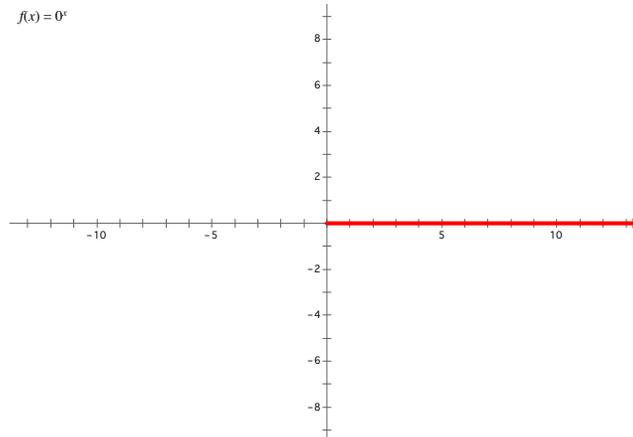
$$f(x) = a \cdot b^x$$

where b and x hold the same properties of the parent function and a is the y intercept. The only difference between this function and its parent function is that the graph of the general form will always pass through (0, a) instead of (0,1).

Mathematical Focus 2

Using 0 as a base for an exponential function would create a graph that violates the properties of exponential functions.

Some students believe that $f(x) = 0^x$ should be an exponential function because it is a real number raised to the power of x , which is the basic format of an exponential function. Intuitively, though, we know that 0 raised to any power will always result in the value of 0. This makes this function a constant function, a vertical line that lies exactly across the x axis. However, because 0 cannot be raised to a negative number (see **Focus 4**), the line begins at $(0,0)$ and goes on along the x axis as $x \rightarrow \infty$:



This graph of this function violates four different properties of exponential functions:

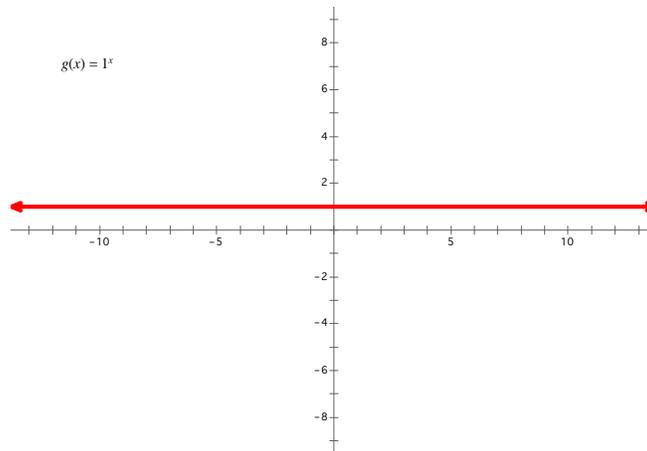
1. This graph does not contain the point $(0,1)$ because the $\lim_{x \rightarrow 0} 0^x$ doesn't exist. This is because:
 - $\lim_{x \rightarrow 0^+} 0^x = 0$
 - $\lim_{x \rightarrow 0^-} 0^x$ is undefined
 - Since the limits are not equal, $\lim_{x \rightarrow 0} 0^x$ does not exist and the value of $f(x)$ at $x = 0$ is not included in the domain.
 - In terms of $f(x) = a \cdot b^x$, the value of $a = 1$. Therefore, the graph should still pass through $(0,1)$, which it does not.
2. $f(x) = 0^x$ is always 0 for its domain and therefore breaks the rule that an exponential functions much be greater than 0 infinity-fold.
3. The domain of $f(x) = 0^x$ does not contain all real numbers because this function is not defined for $x = 0$ (see above) or any negative value (see **Focus 4**).
4. The graph neither increases nor decreases as $x \rightarrow \infty$ because it is a constant function.
5. The function is not one-to-one. Just because $b^x = b^y$, we are not certain that $x = y$.
 - For example, $0^1 = 0^2$, but $1 \neq 2$.

Because these numerous properties are violated, the function $f(x) = 0^x$ cannot be classified as an exponential function.

Mathematical Focus 2

Similar to using 0 as a base, using 1 as a base for an exponential function would create a graph that violates the properties of exponential functions.

Similar to the case of using 0, many students believe that the function $f(x) = 1^x$ is an exponential function because its values can be calculated for any real number. However, 1 raised to any power will result in 1, so it, too, is a constant function:



The following properties are therefore violated:

1. The graph neither increases nor decreases as $x \rightarrow \infty$ because it is a constant function.
2. The function is not one-to-one. Therefore, we cannot be certain that because $b^x = b^y$, $x = y$.
 - a. For example, $1^0 = 1^2$, but $0 \neq 2$.

Because of these violations, $f(x) = 1^x$ is not an exponential function.

Mathematical Focus 3

Using 0 as a base for an exponential function would be undefined for negative values of x .

As shown in the graph in **Focus 2**, the domain of $f(x) = 0^x$ is only defined in the interval $(0, \infty)$. This is because negative values of x would produce a denominator of 0, which makes the value undefined.

For example, $0^{-1} = \frac{1}{0}$, which is undefined.

However, one of the properties of exponential functions states that such functions include the domain of all real numbers. In this case, since the

domain only includes half of real numbers, $f(x) = 0^x$ cannot be classified as an exponential function.

Mathematical Focus 4

Using negative base values would make the function values complex for certain fractional values of x and hinder the function's ability to be a continuous exponential function.

Similar to the cases seen above, many values can be evaluated for functions in the form of exponential functions with negative bases. However, this does not necessarily mean that these functions can be classified as exponential functions. One property of exponential functions that is violated when dealing with negative bases is that the function has the inability to consistently increase or decrease as $x \rightarrow \infty$. Because of properties dealing with the multiplication of negative numbers, these type of functions will constantly alternate between positive and negative values as $x \rightarrow \infty$. For example, using the equation $f(x) = (-2)^x$:

$$\begin{aligned}(-2)^1 &= -2 \\ (-2)^2 &= 4 \\ (-2)^3 &= -8\end{aligned}$$

Another problem we run into as we deal with negative bases is that some negative values raised to certain exponent powers may result in complex values that cannot be graphed on a two-dimensional Cartesian plane. For example, continuing with our example $f(x) = (-2)^x$:

$$(-2)^{1/2} = \sqrt{(-2)} = i\sqrt{2}$$

This value cannot be graphed without having a complex axis. This is also why unlike our scenarios for a base of 0 or 1, $f(x) = (-2)^x$ cannot be graphed.

Because of their inability to consistently increase or decrease and restrictions on the domain, exponential functions cannot have negative bases.

Mathematical Focus 5

Compound interest is a practical application for exponential functions that displays the restrictions on base values.

The most common example for the application of exponential functions is compound interest. The formula for calculating this value is:

$$A = P(1 + r/n)^{nt}$$

where

A = the value after t amount of time

P = the principal amount, which is the initial investment

r = the annual interest rate

n = the number of times interest is compounded per year

t = the number of years

For example, if the principal value invested is \$150 with an annual interest rate of 4% (0.04) that is compounded quarterly (four times a year) for five years, we would have:

$$A = 150(1 + 0.04/4)^{4(5)} = 150(1.01)^{20} \approx 150(1.22) \approx 183.03$$

Therefore, the balance after five years is approximately \$183.03.

In the case of our formula, our base would be $1 + r/n$. The following restrictions apply to this base:

1. $1 + r/n \neq 0$. This would occur if $r/n = -1$. This would occur if either $-r = n$ or $r = -n$. However, neither of these possibilities can occur because neither interest rate nor time can practically be a negative value in a real-life situation.
2. $1 + r/n \neq 1$. This would occur if $r/n = 0$. This would only occur if $r = 0$. However, this would not practically occur because it would produce a constant function, leaving the principal unchanged.
3. $1 + r/n =$ a negative value. This would occur if $r/n < -1$. However, as expressed in the first case, this is not practical in the context of the situation.

Therefore, as our practical case of exponential functions shows, an exponential function cannot have a base of 0, 1, or a negative value.

Mathematical Focus 6

Logarithmic functions, which are directly related to exponential functions and commonly taught at the same time as them, also display similar properties.

Logarithmic functions typically appear in the form:

$$f(x) = \log_b x$$

where, the base b has the same properties:

1. $b \neq 1$
2. $b > 0$

However, there is one more restriction to logarithmic functions that exponential functions do not share: $x > 0$. However, this makes sense intuitively because $b > 0$. Therefore, if b is always positive, so will x . This will make sense after the explanation of the relationship between logarithmic and exponential functions given below.

Logarithmic and exponential functions are directly related because of the fact that:

$$y = \log_b x \Leftrightarrow b^y = x$$

where the first equation is known as **logarithm form** and the second is known as **exponential form**.

Therefore, we can see that x must be positive because an exponential function is always positive. Raising any positive number to some power will always result in a positive number. Many of the other properties hold for both because their forms are interchangeable.